

MANY EXAMPLES OF NON-COCOMPACT FUCHSIAN GROUPS SITTING IN $\mathrm{PSL}_2(\mathbb{Q})$

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ABSTRACT. We construct infinitely many noncommensurable non-cocompact Fuchsian groups Δ of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ so that the set of hyperbolic fixed points of Δ will contain a given finite collection of elements in the boundary of the hyperbolic plane.

1. INTRODUCTION

Let Γ be a Fuchsian group, meaning a discrete subgroup of the group of orientation preserving isometries of H^2 , the hyperbolic plane. A boundary point of H^2 fixed by a parabolic element of a Fuchsian group Γ is referred to as a *cusp* of Γ , and a line fixed by a hyperbolic element is referred to as an *axis* with endpoints called hyperbolic fixed points. For an arbitrary Fuchsian group, determining its set of cusps, hyperbolic fixed points, and/or axes is quite a challenge; for some of the literature addressing this type of problem, see [2] and its references.

Recall that Fuchsian groups Γ_1 and Γ_2 are *commensurable* if Γ_1 has a subgroup of finite index which is conjugate to a subgroup of finite index in Γ_2 . This work has been motivated by the following question(s):

Question(s). If Γ_1 and Γ_2 are finite covolume Fuchsian groups with the same set of cusps (or same set of axes), when are they commensurable?

In [2], Long and Reid exhibit four examples of mutually noncommensurable subgroups of $\mathrm{PSL}_2(\mathbb{Q})$, which are not commensurable with the modular group, but each of them have cusp set exactly $\mathbb{Q} \cup \{\infty\}$; they call such groups *pseudomodular*. It is still unknown whether or not there are infinitely many pseudomodular groups up to commensurability. Any other possible candidate for pseudomodular groups are (non-arithmetic) discrete subgroups $\Delta \leq \mathrm{PSL}_2(\mathbb{Q})$, since their cusp set is contained in $\mathbb{Q} \cup \{\infty\}$. For Fuchsian groups, a boundary point cannot both be a cusp and a hyperbolic fixed point. Hence arithmetic and pseudomodular groups cannot have rational hyperbolic fixed points. So if one can exhibit a hyperbolic element of Δ that has rational fixed

points, then Δ 's cusp set is properly contained in $\mathbb{Q} \cup \{\infty\}$; thus showing Δ to be neither arithmetic nor pseudomodular. So Long and Reid asked, how to predict when rational hyperbolic fixed points are present.

Another motivation arises from a question A. Rapinchuk asked. Are there infinitely many commensurability classes of finite covolume Fuchsian groups sitting in $\mathrm{PSL}_2(\mathbb{Q})$? Vinberg, answering this question in a preprint [5], has introduced a way to produce infinitely many noncommensurable finite covolume Fuchsian groups in $\mathrm{SL}_2(\mathbb{Q})$. His examples arise as the even subgroup of a group generated by reflections in the sides of quadrilaterals. To establish that he constructed infinitely many groups up to commensurability, Vinberg uses results (from [4]) about the least ring of definition for his examples.

This paper provides a new solution to Rapinchuk's question, which is different from Vinberg's, and the results also address the presence of rational hyperbolic fixed points. Namely, we will construct Fuchsian groups sitting in $\mathrm{PSL}_2(\mathbb{Q})$, all of which will possess a given finite set of rational hyperbolic fixed points. The main result is

Theorem. *Let Y be a finite set of rational boundary points of the hyperbolic plane. Then there are infinitely many noncommensurable finite covolume Fuchsian groups sitting in $\mathrm{PSL}_2(\mathbb{Q})$, whose set of hyperbolic fixed points contains Y .*

This will be proved in section 4. As a brief outline, let Y be a finite number of boundary points of the hyperbolic plane. We construct (with considerable freedom) examples of Fuchsian groups Γ of signature $(0 : 2, \dots, 2; 1; 0)$ such that the set of hyperbolic fixed points of Γ contains Y (see section 2); furthermore, when Y is a set of rational boundary points, and by restricting some of the freedom in the construction, one can guarantee $\Gamma \leq \mathrm{PGL}_2(\mathbb{Q})$. Then we address when the constructed groups are mutually noncommensurable, which relies on analyzing how they act on different trees (see section 3). Specifically, we consider fixed points of the action of Γ on Serre's trees of $\mathrm{SL}_2(\mathbb{Q}_p)$ for primes $p \equiv 3 \pmod{4}$ (see Proposition 3). This perspective allows us to construct an infinite family of mutually noncommensurable groups in $\mathrm{PSL}_2(\mathbb{Q})$.

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2. THE CONSTRUCTION

In this section, we will construct a fundamental domain for a Fuchsian group Γ of signature $(0 : 2, \dots, 2; 1; 0)$ so that the set of hyperbolic fixed points of Γ , notated by $\mathrm{HFix}(\Gamma)$, will contain a given finite set Y of elements in boundary of the hyperbolic plane. Before the construction begins, we introduce some notation and a lemma.

Let H^2 be the hyperbolic plane. We will write \overline{xy} for the closure of the geodesic line with end points x, y in the closure of H^2 . The isometry denoted by ρ_f is rotation by π with fixed point $f \in H^2$. Let v_0 and v_∞ be distinct elements in ∂H^2 ; then we have an order \leq on $\partial H^2 \setminus \{v_\infty\}$ (i.e. the order on the real line in the upper half plane model).

Lemma 2.1. *Let $x < v < y < u$ in $\partial H^2 \setminus \{v_\infty\}$. For each f in the interior of \overline{cy} , where $c = \overline{vu} \cap \overline{xy}$, one constructs $t = \rho_f(u)$ and $w = \rho_f(v)$. Then $x < v < t < y < w < u$ in ∂H^2 , $f = \overline{xy} \cap \overline{tu}$, and $f \in \overline{vw}$.*

We omit the proof, but include a figure to help illustrate the lemma.

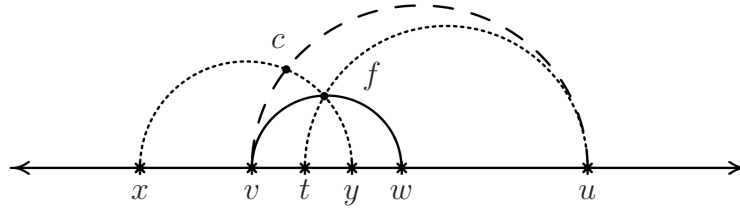


FIGURE 1. Lemma 2.1 drawn in upper half plane model.

2.1. The Construction of a Fundamental Domain for Γ . As an overview of the construction (Figure 2 illustrates an example), we start with a finite set of points in the boundary of the hyperbolic plane (the set $Y = \{y_i\}$). We use Lemma 2.1 to sequentially construct the vertices and edges (the solid lines in Figure 2) of an ideal convex polygon in H^2 . We then show that the ideal convex polygon constructed is a fundamental domain for a discrete group Γ generated by isometries that are rotations by π . Furthermore, Γ is guaranteed to possess a set of hyperbolic elements (the dash lines are their axes in Figure 2) whose fixed points contain the initially given set of boundary points.

Below we will have the notational convention: $f_i \in H^2$, $\rho_i = \rho_{f_i}$ is rotation by π with fixed point f_i , and $y_i, x_i, v_i \in \partial H^2$. As a slight variation when considering the hyperboloid model of the hyperbolic plane (see subsection 2.2), f_i, x_i, y_i, v_i will be vectors in Minkowski space $\mathbb{R}^{2,1}$.

We begin the construction; let Y be a finite set of $n - 1$ points in the boundary of the hyperbolic plane, and let $Y = \{y_i\}$ so that $v_0 < y_1 < \dots < y_{n-1} \neq v_\infty$ in the boundary of the hyperbolic plane.

1st Step: Choose x_1 such that $v_0 < x_1 < y_1$ in ∂H^2 , and then choose $f_1 \in \overline{x_1 y_1}$. Define $v_1 = \rho_1(v_0)$; note $v_0 < x_1 < v_1 < y_1$ in ∂H^2 , and $f_1 \in \overline{v_0 v_1}$.

When $n > 2$, let $i \in \{2, \dots, n - 1\}$.

i^{th} Step: Let $x_{i-1} < v_{i-1} < y_{i-1} < y_i$ in ∂H^2 . By Lemma 2.1, one can choose a $f_i \in \overline{x_{i-1} y_{i-1}}$ and construct $x_i = \rho_i(y_i)$ and $v_i = \rho_i(v_{i-1})$, so that $v_{i-1} < x_i < y_{i-1} < v_i < y_i$ and $f_i \in \overline{v_{i-1} v_i}$.

n^{th} Step: Let $x_{n-1} < v_{n-1} < y_{n-1} \neq v_\infty$ in ∂H^2 . Now construct $f_n = \overline{x_{n-1} y_{n-1}} \cap \overline{v_{n-1} v_\infty}$, and define $v_n = \rho_n(v_{n-1})$.

Last Step: Given $\rho_n \dots \rho_1(v_0) = v_n$, construct $f_0 \in \overline{v_0 v_n}$ and ρ_0 so that $\rho_n \dots \rho_1 \rho_0$ is parabolic fixing v_n .

Remark. When this construction is done with vectors in Minkowski space $\mathbb{R}^{2,1}$, v_n and v_∞ are linearly dependent light-like vectors, and in the last step one can see for $f_0 \in \text{span}\{v_n + v_0\}$, the element ρ_0 as a Lorentz transformation maps v_n to v_0 (as vectors in $\mathbb{R}^{2,1}$), which shows $\rho_n \dots \rho_1 \rho_0$ is parabolic fixing v_n .

We have an ideal $n + 1$ sided convex polygon, P , with vertices $\{v_0, v_1, \dots, v_{n-1}, v_n = v_\infty\}$; furthermore, f_i is on the edge $\overline{v_{i-1} v_i}$, and f_0 is on the edge $\overline{v_n v_0}$ (see Figure 2). Since ρ_i is rotation by π with fixed point $f_i \in H^2$, ρ_i maps $\overline{v_{i-1} v_i}$ to itself (likewise, ρ_0 maps $\overline{v_n v_0}$ to itself); that is, ρ_i maps the directed edge $\overline{f_i v_{i-1}}$ to the directed edge $\overline{f_i v_i}$, and ρ_0 maps the directed edge $\overline{f_0 v_0}$ to the directed edge $\overline{f_0 v_n}$. By Poincaré's Theorem (see section §9.8 in [1]), the group Γ generated by $\{\rho_1, \dots, \rho_n, \rho_0\}$ is discrete, and P is a fundamental domain for Γ . In the last step, we made $\rho_n \dots \rho_1 \rho_0$ parabolic fixing v_n ; thus H^2/Γ is a complete finite area once punctured 2-sphere with $n + 1$ cone points of order 2.

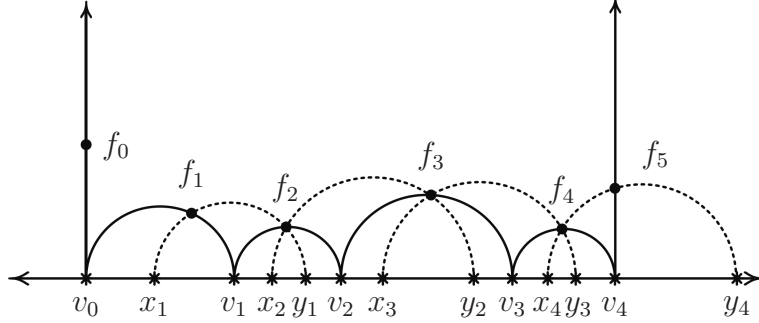


FIGURE 2. The solid lines bound a fundamental domain for Γ in the upper half plane model with $Y = \{y_1, \dots, y_4\}$.

For $1 \leq i < n$, the element $\rho_i \rho_{i+1}$ is hyperbolic with axis $\overline{x_i y_i}$, since f_i and f_{i+1} both lie on the geodesic line $\overline{x_i y_i}$ (by construction); therefore, y_i is a hyperbolic fixed point for $\rho_i \rho_{i+1} \in \Gamma$.

By Lemma 2.1, one sees that there are infinitely many choices for each f_i (for $1 \leq i < n$), producing infinitely many such Fuchsian groups Γ ; establishing the following:

Proposition 1. *Let Y be a finite set of $n-1$ points in ∂H^2 . Then there are infinitely many Fuchsian groups Γ of finite covolume of signature $(0 : \underbrace{2, \dots, 2}_{n+1}; 1; 0)$ such that $Y \subset \mathrm{HFix}(\Gamma)$.*

2.2. Γ in $\mathrm{O}^+(2, 1)$ and $\mathrm{PGL}_2(\mathbb{R})$. Let V be a dimension 3 real vector space with a nondegenerate quadratic form $\langle \bullet, \bullet \rangle$ of signature $(2, 1)$. Choose a basis (e_1, e_2, e_0) with $\langle e_i, e_j \rangle = 0$ if $i \neq j$, $\langle e_i, e_i \rangle = 1$ if $i \geq 1$, and $\langle e_0, e_0 \rangle = -1$. Such a basis is called a Lorentz orthonormal basis, and V is denoted as $\mathbb{R}^{2,1}$ when such a basis is fixed; $\mathbb{R}^{2,1}$ is called Minkowski space. We will notate $L = \{v : \langle v, v \rangle = 0\}$ (the set of light-like vectors) and $T = \{v : \langle v, v \rangle < 0\}$ (the set of time-like vectors). Let L^+ and T^+ be the sets of vectors with positive e_0 -coordinate in L and T , respectively. We let $\mathrm{O}^+(2, 1)$ be the group of linear transformations of V that preserve the quadratic form and upper sheet of the hyperboloid $\mathcal{H} = \{v : \langle v, v \rangle = -1\} \cap T^+$. With a Lorentz orthonormal basis $\{e_1, e_2, e_0\}$, let the \mathbb{Q} -linear combination of $\{e_1, e_2, e_0\}$ be denoted by $\mathbb{Q}^{2,1}$; furthermore, let $L_{\mathbb{Q}}^+ = L^+ \cap \mathbb{Q}^{2,1}$ and $T_{\mathbb{Q}}^+ = T^+ \cap \mathbb{Q}^{2,1}$.

For every $v \in T^+$, there is a 2 by 2 real symmetric matrix whose determinant is $-\langle v, v \rangle$; namely,

$$v \mapsto \begin{pmatrix} \langle v, e_2 + e_0 \rangle & \langle v, e_1 \rangle \\ \langle v, e_1 \rangle & \langle v, -e_2 + e_0 \rangle \end{pmatrix}.$$

We have that $\mathrm{GL}_2(\mathbb{R})$ acts on 2 by 2 real symmetric matrices by similarity; that is, $\Sigma \mapsto M^t \Sigma M$, where $M \in \mathrm{GL}_2(\mathbb{R})$ and Σ is a 2 by 2 real symmetric matrix. We use this to relate the hyperboloid and upper half plane models.

Example. Consider the isometry ρ_f , rotation by π with fixed point $f \in H^2$.

In the hyperboloid model, ρ_f corresponds to an element in $\mathrm{O}^+(2, 1)$; let $f \in T^+$ be fixed by $\rho_f \in \mathrm{O}^+(2, 1)$. Let Σ_f be the 2 by 2 real symmetric matrix associated to f ,

$$\Sigma_f = \begin{pmatrix} \langle f, e_2 + e_0 \rangle & \langle f, e_1 \rangle \\ \langle f, e_1 \rangle & \langle f, -e_2 + e_0 \rangle \end{pmatrix}.$$

In the upper half plane model of H^2 , say $a + bi$ is fixed by ρ_f as a matrix in $\mathrm{SL}_2(\mathbb{R})$; that is,

$$(*) \quad \rho_f = \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{b} \begin{pmatrix} a & -(b^2 + a^2) \\ 1 & -a \end{pmatrix}.$$

We have that ρ_f , as the matrix from (*), acts by similarity on 2 by 2 real symmetric matrices and fixes Σ_f when

$$b = \frac{\sqrt{|\langle f, f \rangle|}}{\langle f, e_2 + e_0 \rangle} \quad \text{and} \quad a = -\frac{\langle f, e_1 \rangle}{\langle f, e_2 + e_0 \rangle}.$$

Note: in $\mathrm{PGL}_2(\mathbb{R})$, ρ_f can be represented by a matrix with entries in \mathbb{Q} when $\langle f, f \rangle, \langle f, e_i \rangle$ are all in \mathbb{Q} .

Proposition 2. *Let $v_0 < y_1 < \dots < y_{n-1} \neq v_\infty$ in $L_{\mathbb{Q}}^+$. Then there are infinitely many non-cocompact Fuchsian groups Δ of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ such that $\{y_1, \dots, y_{n-1}\} \subset \mathrm{HFix}(\Delta)$.*

Proof. We follow the construction of Γ in Minkowski space. In the i^{th} step (for $1 \leq i < n$) of subsection 2.1, we additionally require the choice of x_1 and the f_i , as vectors in Minkowski space, to lie in $\mathbb{Q}^{(2,1)}$. Furthermore, f_n and f_0 will be in $\mathbb{Q}^{(2,1)}$, since all x_i, v_i, y_i will be in $\mathbb{Q}^{(2,1)}$. Thus Γ will sit in $\mathrm{PGL}_2(\mathbb{Q})$. Even with these additional requirements in the construction, there are still infinitely many choices for each f_i (for $1 \leq i < n$), producing infinitely many such Fuchsian groups Γ .

For each Γ , $\{y_i\} \subset \mathrm{HFix}(\Gamma)$, and let Δ be the kernel of $\Gamma \longrightarrow \mathrm{PGL}_2(\mathbb{Q})/\mathrm{PSL}_2(\mathbb{Q})$, which is of finite index in Γ ; therefore, it follows $\{y_i\} \subset \mathrm{HFix}(\Delta)$. \square

3. ACTING ON THE TREE OF $\mathrm{SL}_2(\mathbb{Q}_p)$

As in Serre's book [3], let K denotes a field with a discrete valuation v ; recall that v is a homomorphism of K^\times onto \mathbb{Z} , and \mathcal{O}_v denotes the valuation ring of K , i.e. the set of $x \in K$ such that $v(x) \geq 0$ or $x = 0$. Fix an element $\pi \in K$ with $v(\pi) = 1$, the uniformizer. If $K = \mathbb{Q}$, then most v subscripts are replaced with the letter p for the p -adic valuation v_p .

Let V be a vector space of dimension 2 over K . A lattice in V is any finitely generated \mathcal{O}_v -submodule of V which generates the K -vector space V ; such a module is free of rank 2. The group K^\times acts on the set of lattices; we call the orbit of a lattice L under this action its class (at times notated $[L] = \Lambda$), and two lattices belonging to the same class are called equivalent. The set of lattice classes is denoted by \mathcal{T}_v , which is made into a combinatorial graph with edges between Λ_0 and Λ_1 when $[L_i] = \Lambda_i$ such that $L_0 \leq L_1$ and $L_1/L_0 \cong \mathcal{O}_v/\pi\mathcal{O}_v$. Serre proved that \mathcal{T}_v is a tree.

Lemma 3.1. *Let Γ be generated by a finite number of ρ_j , where $\rho_j = C_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C_j^{-1}$ with $C_j \in \mathrm{GL}_2(K)$.*

Then the following are equivalent statements about the action of Γ on the tree \mathcal{T}_v :

- (1) $\Gamma \leq \mathrm{Stab}(\Lambda)$ for some $\Lambda \in \mathcal{T}_v$;
- (2) $\bigcap C_j \mathrm{Fix}_{\mathcal{T}_v} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \neq \emptyset$;
- (3) For each pair (m, k) , $\mathrm{Fix}_{\mathcal{T}_v}(\rho_m \rho_k) \neq \emptyset$.

Proof. (1) \Leftrightarrow (2): follows from the equality

$$\mathrm{Fix}_{\mathcal{T}_v} \left(C_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C_j^{-1} \right) = C_j \mathrm{Fix}_{\mathcal{T}_v} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right).$$

(3) \Leftrightarrow (1): see [3], I§6.5. \square

Lemma 3.2. *Let Γ be generated by a finite number of ρ_j , where $\rho_j = C_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C_j^{-1}$ with $C_j = \begin{pmatrix} b_j & a_j \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K)$, as in (*).*

When -1 is not a square in K , the following are equivalent:

- (1) $\Gamma \leq \mathrm{Stab}(\Lambda)$ for some $\Lambda \in \mathcal{T}_v$;
- (2) For each pair (m, k) ,

$$C_k^{-1} C_m \in \mathrm{GL}_2(\mathcal{O}_v);$$

- (3) For each pair (m, k) ,

$$v(a_m - a_k) \geq v(b_m) = v(b_k).$$

Proof. When -1 is not a square in K , we have that $\text{Fix}_{\mathcal{T}_v}((\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})) = \{[\mathcal{O}_v^2]\}$, where \mathcal{O}_v^2 is the standard lattice.

Let ρ_m and ρ_k be two generators of Γ . Then $\text{Fix}_{\mathcal{T}_v}(\rho_m \rho_k) \neq \emptyset$ if and only if

$$C_m \text{Fix}_{\mathcal{T}_v}((\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})) \cap C_k \text{Fix}_{\mathcal{T}_v}((\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})) \neq \emptyset,$$

and that holds only when $C_k^{-1}C_m \in \text{GL}_2(\mathcal{O}_v)$. By Lemma 3.1, (1) and (2) are equivalent.

To complete the proof note that $C_k^{-1}C_m = \begin{pmatrix} \frac{b_m}{b_k} & \frac{a_m - a_k}{b_k} \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathcal{O}_v)$ if and only if

$$v\left(\frac{b_m}{b_k}\right) = 0 \quad \text{and} \quad v\left(\frac{a_m - a_k}{b_k}\right) \geq 0.$$

□

Proposition 3. *Let $v_0 < y_1 < y_2 < \dots < y_{n-1} \neq v_\infty$ in $L_{\mathbb{Q}}^+$ ($n > 2$), and let a prime $p \equiv 3 \pmod{4}$. Then there are non-cocompact Fuchsian groups Δ of finite covolume sitting in $\text{PSL}_2(\mathbb{Q})$ with $\{y_1, \dots, y_{n-1}\} \subset \text{HFix}(\Delta)$, each of which stabilize no vertex in the tree \mathcal{T}_p .*

Proof. Consider the construction of Γ in subsection 2.1 in Minkowski space; below we will describe additional requirements for choosing x_1 and the f_i .

In the 1st step, choose $x_1 \in L_{\mathbb{Q}}^+$ so $v_0 < x_1 < y_1$. When choosing f_1 , additionally require that $f_1 \in \text{span}_{\mathbb{Q}}\{x_1, y_1\}$ and $|\langle f_1, f_1 \rangle|$ is in the rational square class $p(\mathbb{Q}^\times)^2 = \{p\alpha^2 : \alpha \in \mathbb{Q}\}$, which is possible because $\text{span}_{\mathbb{Q}}\{x_1, y_1\}$ is isotropic. In the i^{th} step (for $1 < i < n$), one specifies a rational square class, say $n_i(\mathbb{Q}^\times)^2 \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ (n_i a square free integer) such that $p \nmid n_i$. When choosing f_i , additionally require that $f_i \in \text{span}_{\mathbb{Q}}\{x_{i-1}, y_{i-1}\}$ and $|\langle f_i, f_i \rangle| \in n_i(\mathbb{Q}^\times)^2$, which is also possible because $\text{span}_{\mathbb{Q}}\{x_{i-1}, y_{i-1}\}$ is isotropic.

Now note (as in the example in subsection 2.2) that ρ_{f_i} as an element of $\text{PGL}_2(\mathbb{Q})$ is given by the matrix

$$\rho_{f_i} = \begin{pmatrix} b_i & a_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_i & a_i \\ 0 & 1 \end{pmatrix}^{-1},$$

where $b_i = \frac{\sqrt{|\langle f_i, f_i \rangle|}}{\langle f_i, e_2 + e_0 \rangle}$ and $a_i = -\frac{\langle f_i, e_1 \rangle}{\langle f_i, e_2 + e_0 \rangle}$.

Since b_1 has a factor \sqrt{p} , and each b_i ($1 < i < n$) does not,

$$v_p(b_i) \neq v_p(b_1).$$

Moreover, -1 is not a square in \mathbb{Q}_p (since $p \equiv 3 \pmod{4}$). By Lemma 3.2, Γ stabilizes no vertex in \mathcal{T}_p , and by construction $\{y_i\} \subset \text{HFix}(\Gamma)$. Now

let Δ be the kernel of $\Gamma \rightarrow \mathrm{PGL}_2(\mathbb{Q})/\mathrm{PSL}_2(\mathbb{Q})$, which is of finite index in Γ ; then Δ also stabilizes no vertex in \mathcal{T}_p and $\{y_i\} \subset \mathrm{HFix}(\Delta)$. \square

Remark. For each Δ constructed in the proof of Proposition 2, there is an integer m such that Δ stabilizes a vertex of \mathcal{T}_q for all primes $q > m$. To see this, choose m large enough so that m is greater than all the denominators of the entries of a matrix representing ρ_{f_i} , for each i , as an element of $\mathrm{PGL}_2(\mathbb{Q})$

4. PROOF OF THE THEOREM

Theorem. *Let Y be a finite set of rational points in the boundary of the hyperbolic plane. Then there are infinitely many noncommensurable non-cocompact Fuchsian groups Δ of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ so that $Y \subset \mathrm{HFix}(\Delta)$.*

Proof. We can let Y be a finite set of two or more rational points in the boundary of the hyperbolic plane (just add points if fewer than 2 are given). Let $Y = \{y_i\}$, so that $v_0 < y_1 < y_2 < \cdots < y_{n-1} \neq v_\infty$ in $L_{\mathbb{Q}}^+$ ($n > 2$). Now let the family $\{\Delta\}$ be the set of non-cocompact Fuchsian groups of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ such that $\{y_1, \dots, y_{n-1}\} \subset \mathrm{HFix}(\Delta)$, which are constructed in the proof of Proposition 2.

Assume for the purpose of contradiction that there is a finite number k of commensurability classes in family $\{\Delta\}$. Let $\{\Delta_1, \dots, \Delta_k\}$ be distinct representatives from the k commensurability classes.

From the remark just after Proposition 3, there is an integer m such that each Δ_j ($1 \leq j \leq k$) stabilizes a vertex in \mathcal{T}_q for all $q > m$. By Dirichlet's theorem on arithmetic progressions, we can choose a prime $p > m$ and $p \equiv 3 \pmod{4}$. By Proposition 3, there is $\Delta_{k+1} \in \{\Delta\}$ with $\{y_i\} \subset \mathrm{HFix}(\Delta_{k+1})$ and so that Δ_{k+1} does not stabilize any vertex of \mathcal{T}_p . Therefore, each Δ_j ($1 \leq j \leq k$) stabilizes a vertex in \mathcal{T}_p but Δ_{k+1} does not. For subgroups of $\mathrm{PSL}_2(\mathbb{Q})$, the presence or absence of fixed points in \mathcal{T}_p descends to finite index subgroups, and is invariant under conjugation. Thus Δ_{k+1} is not commensurable with any of the Δ_j ($1 \leq j \leq k$), which contradicts the assumption there are a finite number of commensurability classes in family $\{\Delta\}$. \square

Remark. By using Proposition 3 and the remark just after it, one can inductively construct an infinite family where the members lie in different commensurability classes.

As mentioned in the introduction, a boundary point cannot both be a cusp and a hyperbolic fixed point, for Fuchsian groups; thus a direct corollary of the theorem is

Corollary. *Let Y be finite set of rationals. Then there are infinitely many noncommensurable non-cocompact Fuchsian groups of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ whose cusp set is properly contained in $(\mathbb{Q} \setminus Y) \cup \{\infty\}$.*

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